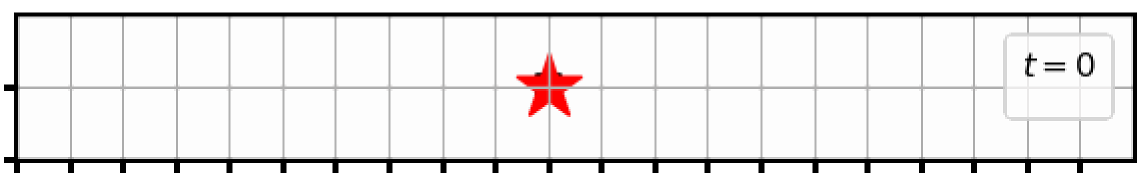
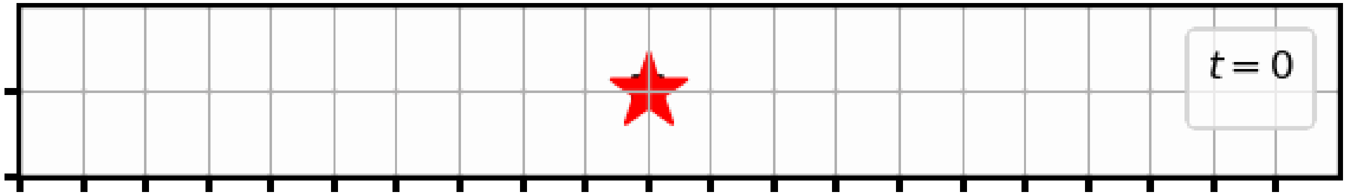
Visitation statistics of 1d random walks

Léo Régnier
(with M. Dolgushev, S. Redner and O. Bénichou)





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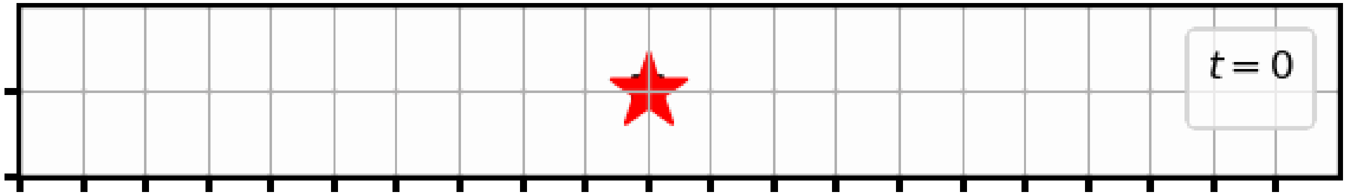

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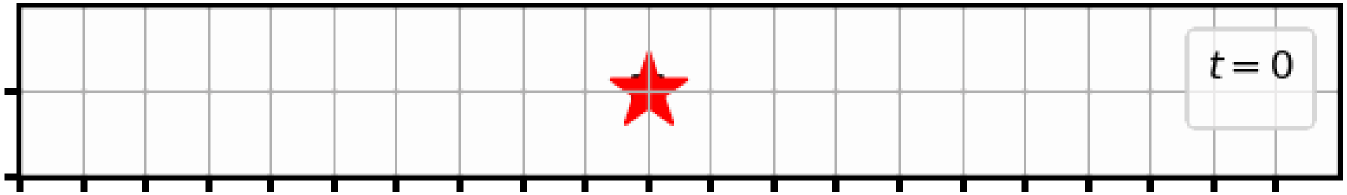

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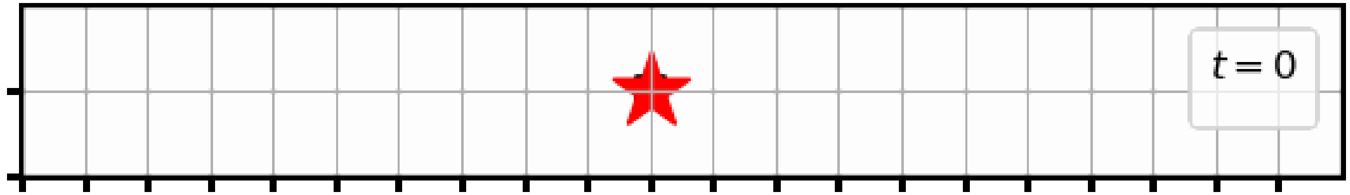

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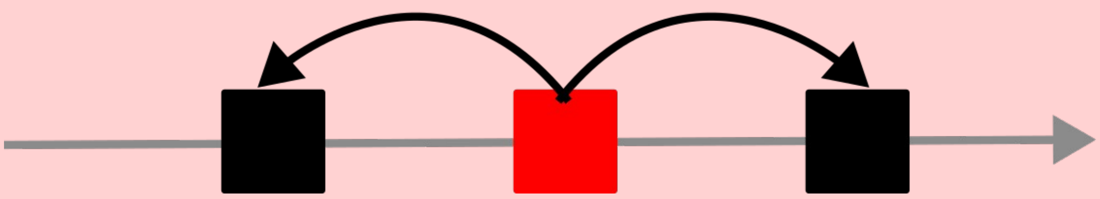
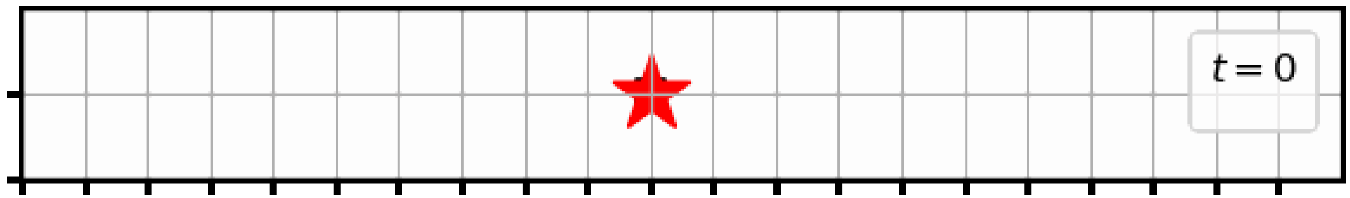
The jumps distribution is...

- **Symmetric,**

$p(\ell_i = 1) = 1/2$





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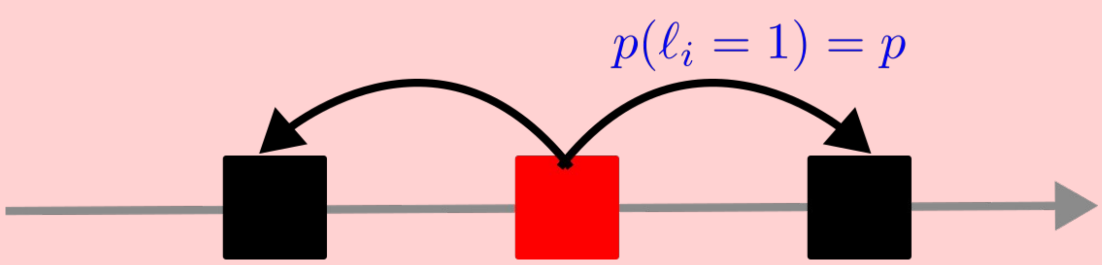

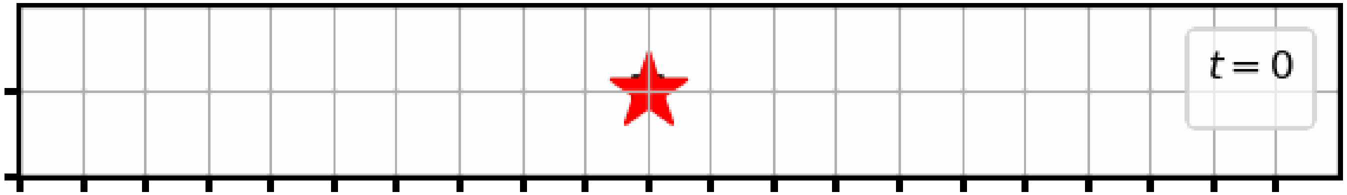

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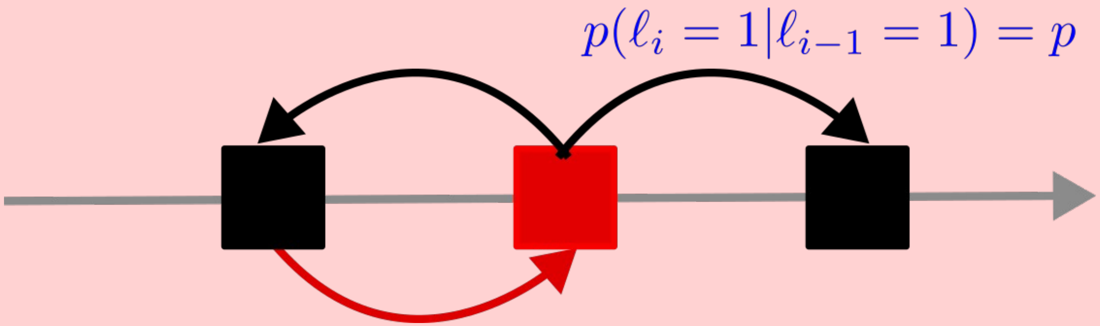
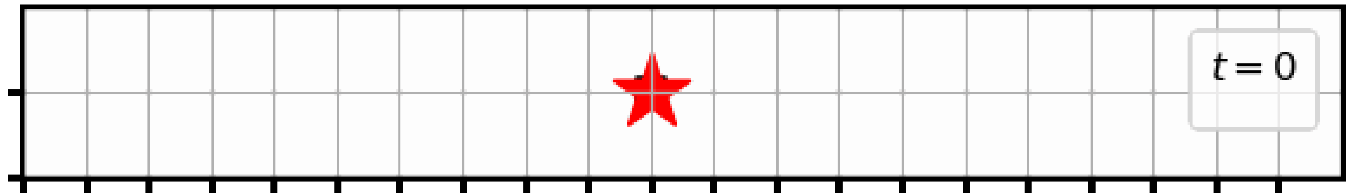
- **Correlated,**



The random walk explores the 1d line up to a time t.



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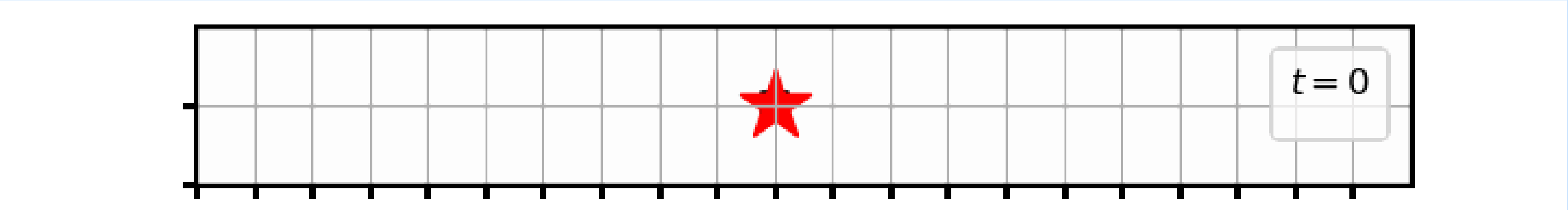

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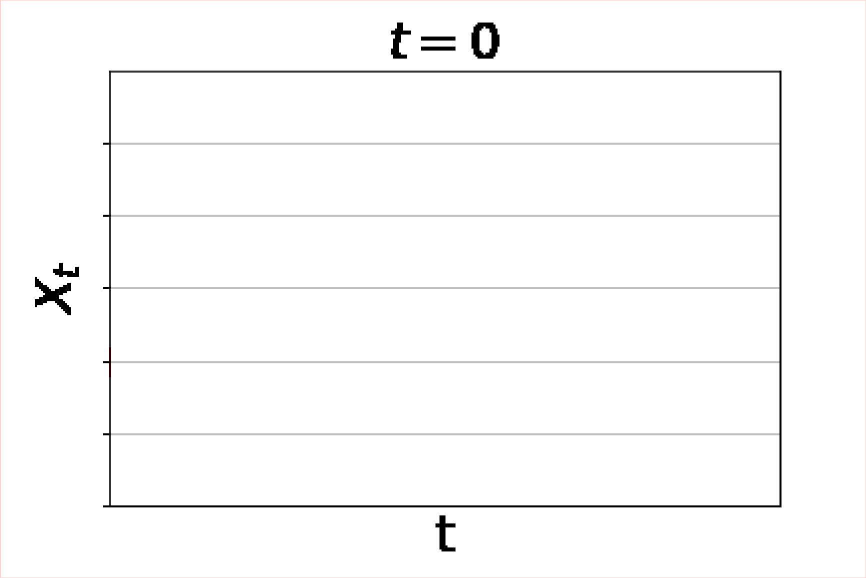

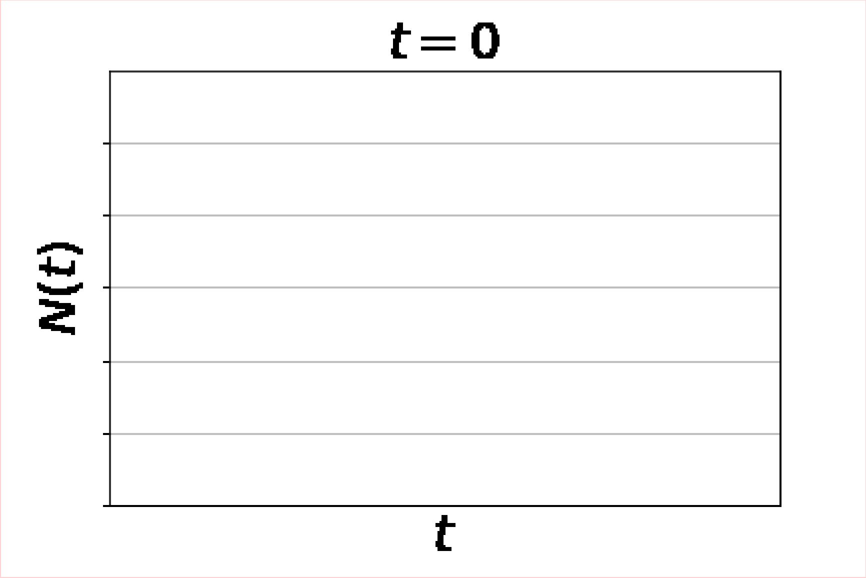

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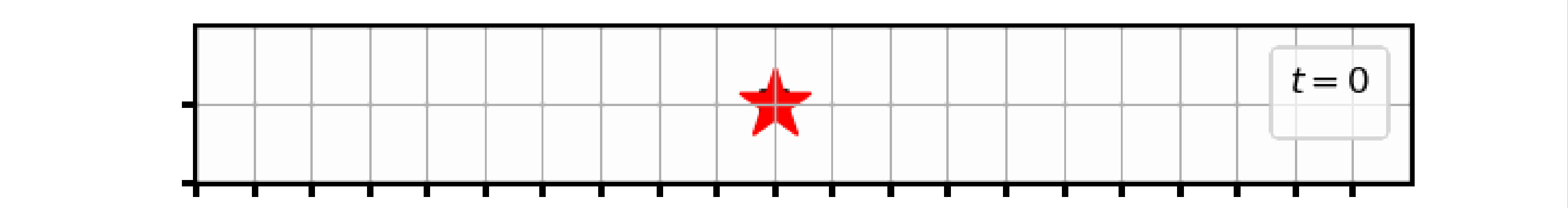

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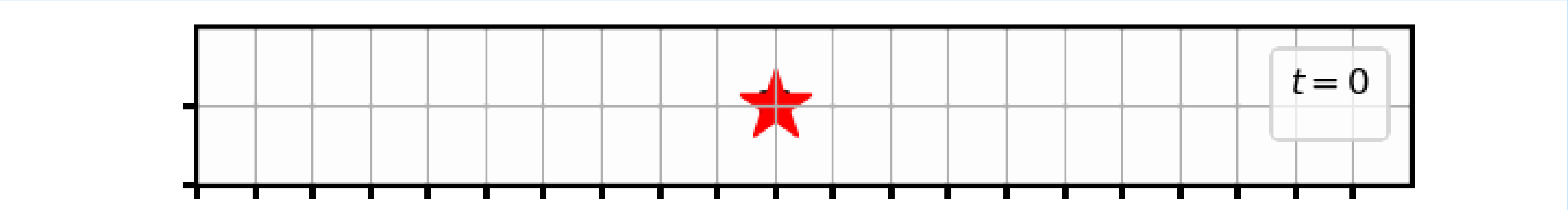

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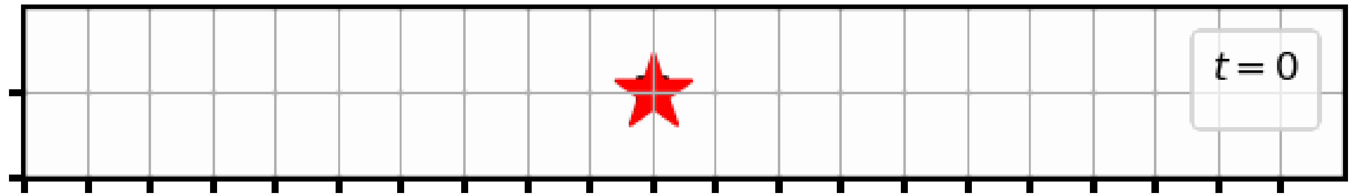

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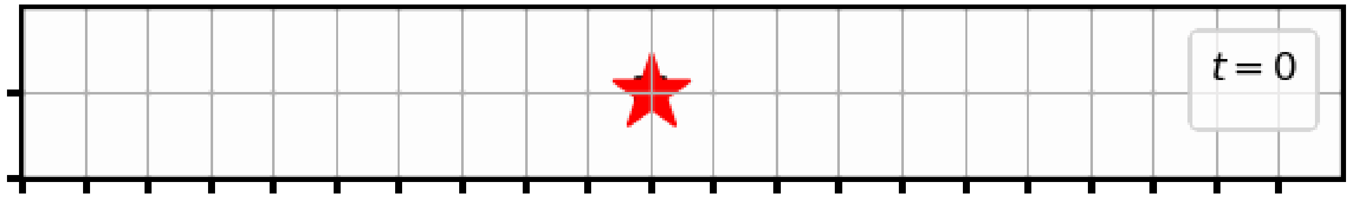

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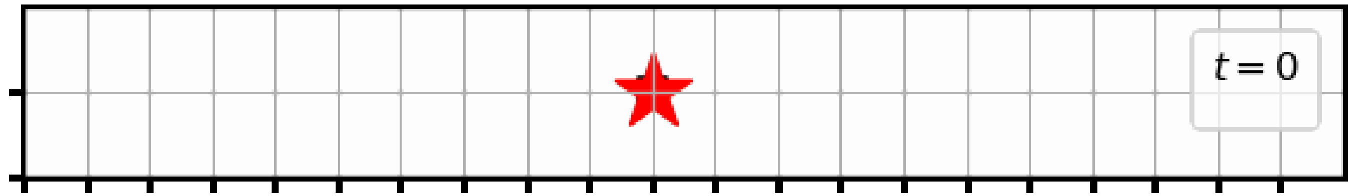

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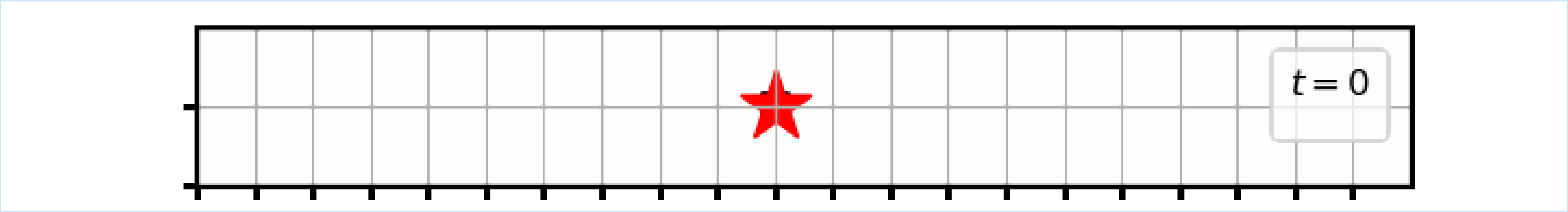

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Partial characterization of the exploration process!

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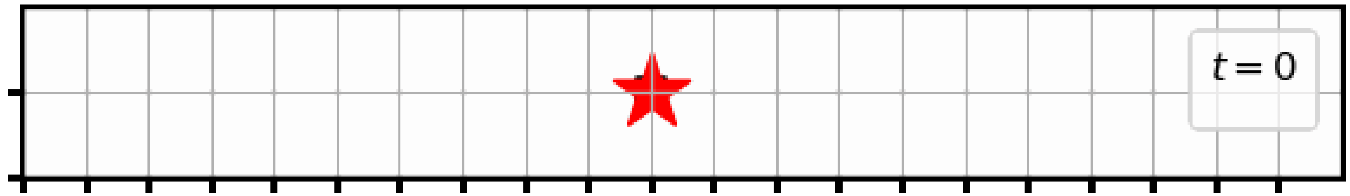

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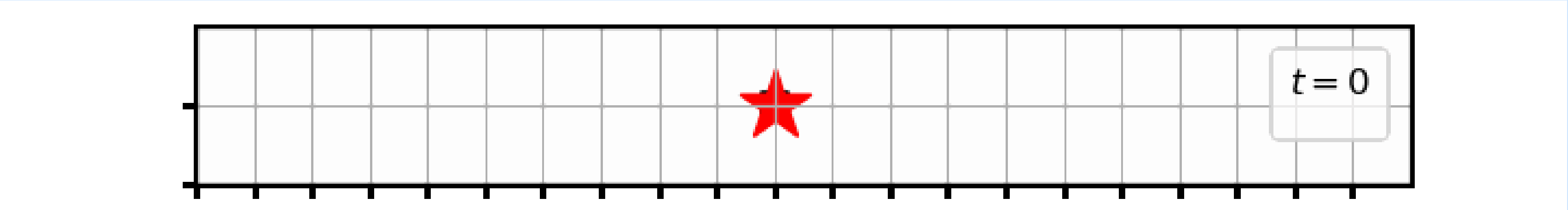

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By its muliple time points distribution...

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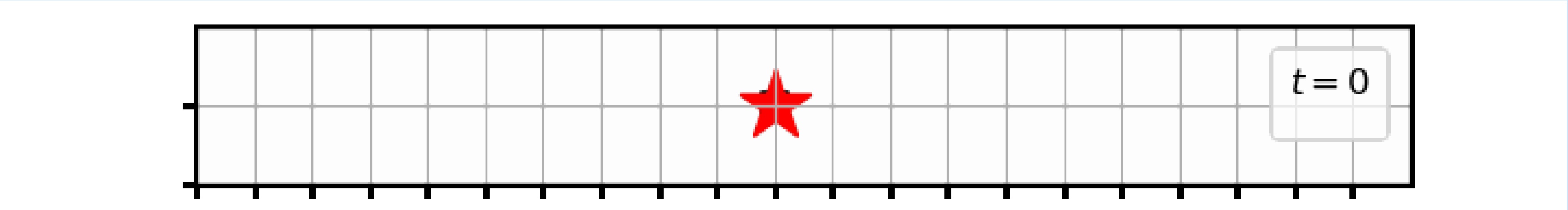

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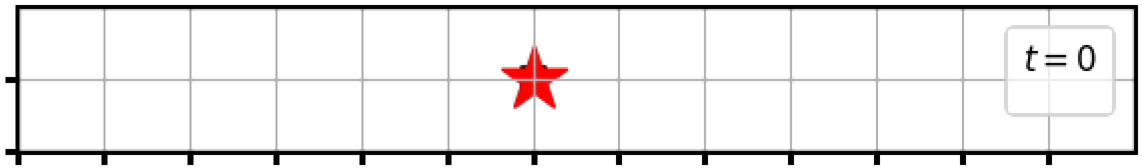

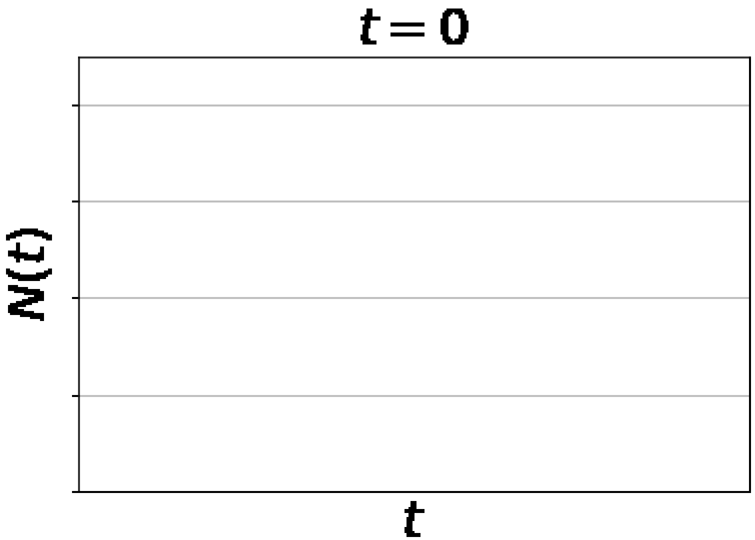

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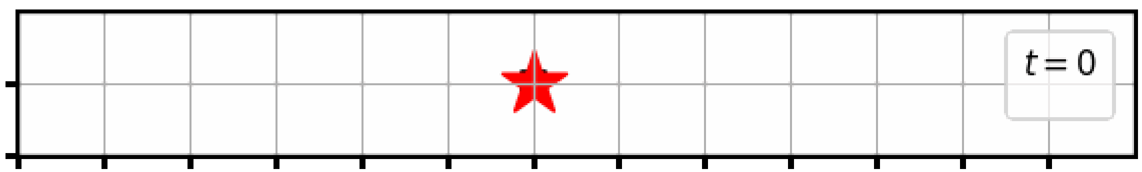

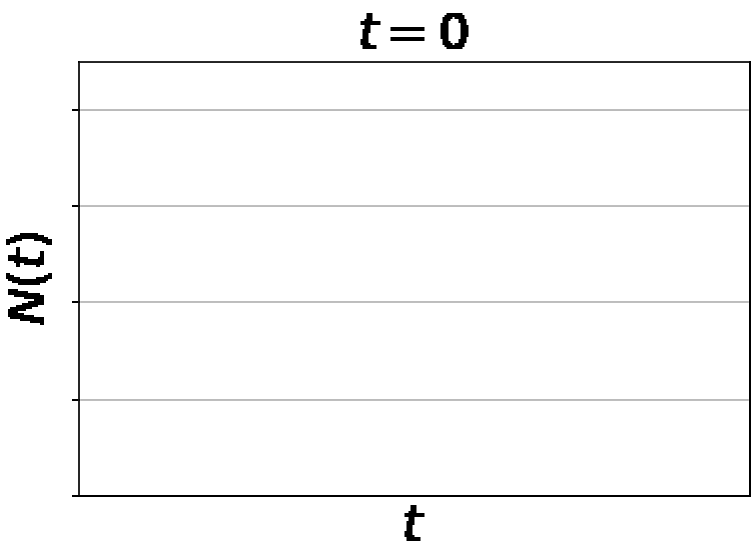

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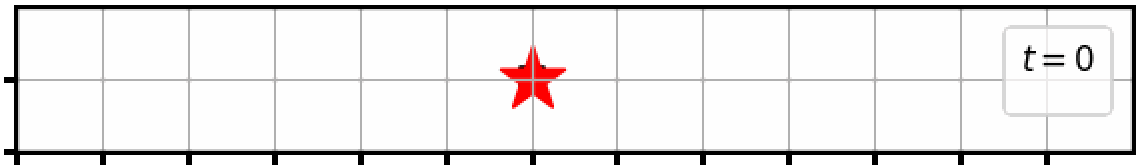

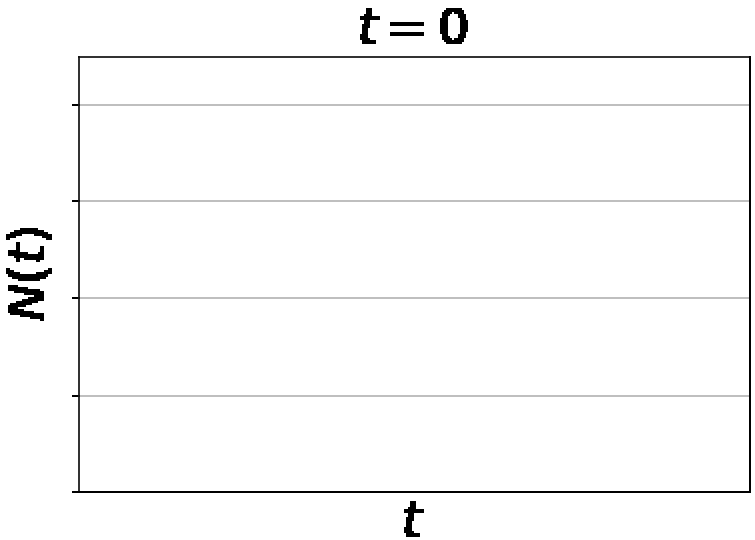

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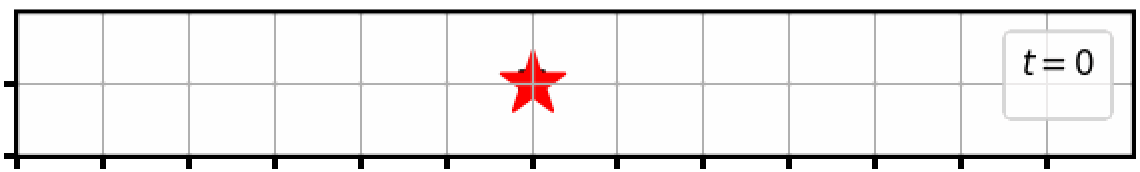

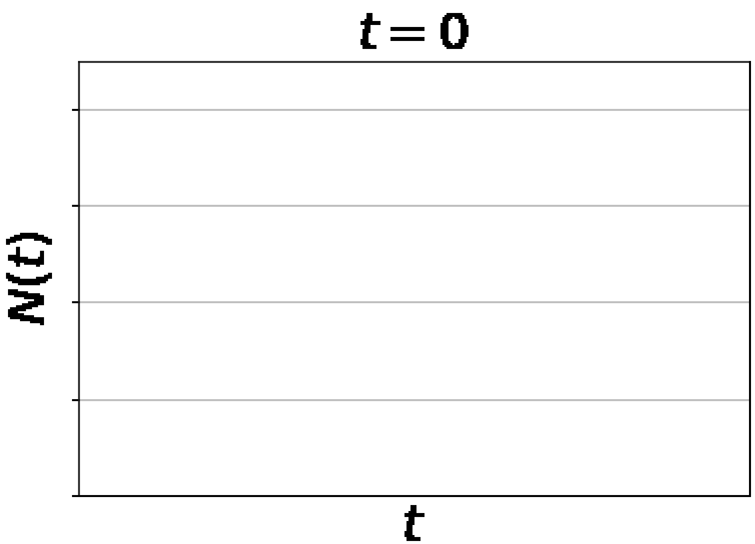

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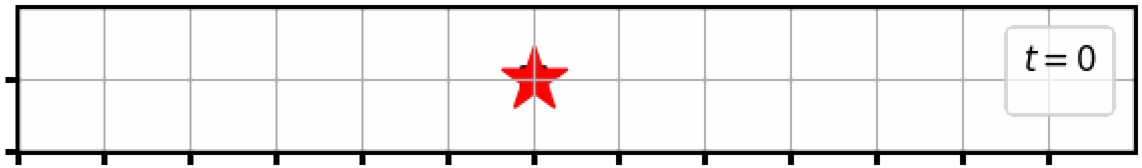

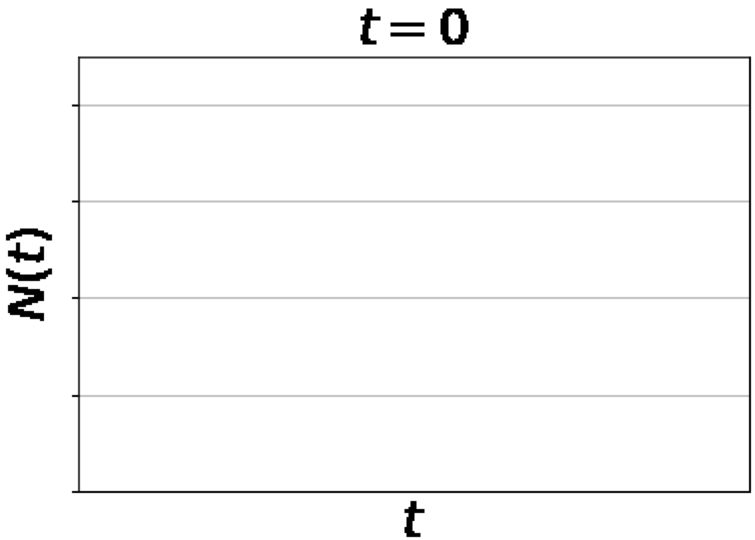

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They are independent (up to a conditionning on the positions for the biased walks)

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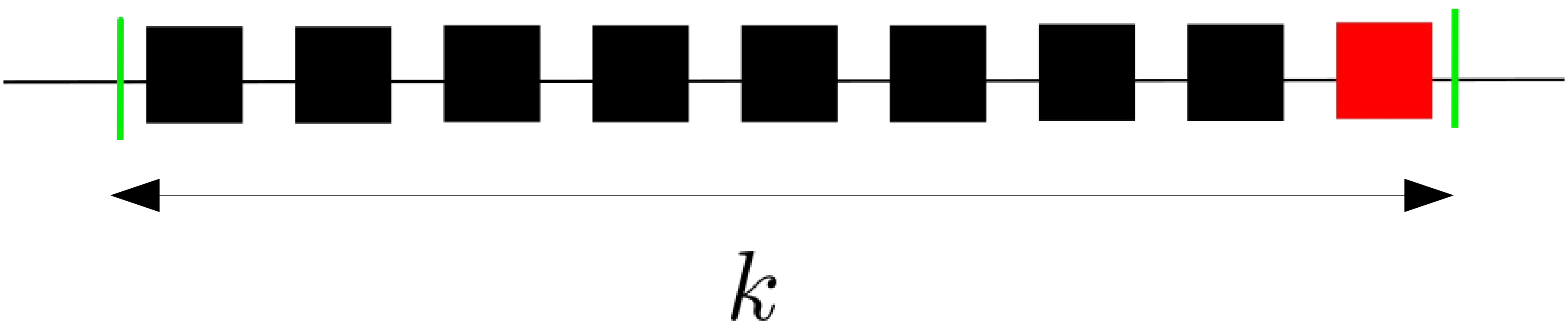
We start by giving the general steps to obtain the **cumulative two times distribution**, for a symmetric walk,

$$\mathbb{P}(N(t_1) \geq n_1; N(t_2) \geq n_2)$$

Step 1: We start by solving the exit time problem from an interval:

Once a new site is visited, we start at one end of the visited domain of length k.



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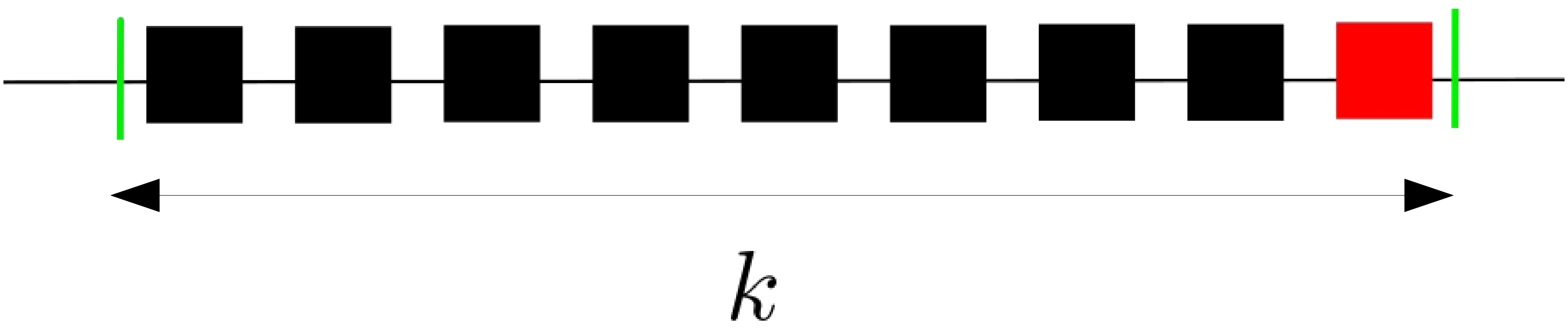
The first exit time is τ_k.

$$\boxed{\mathbb{P}(\tau_k = t) = F(t,k)}$$



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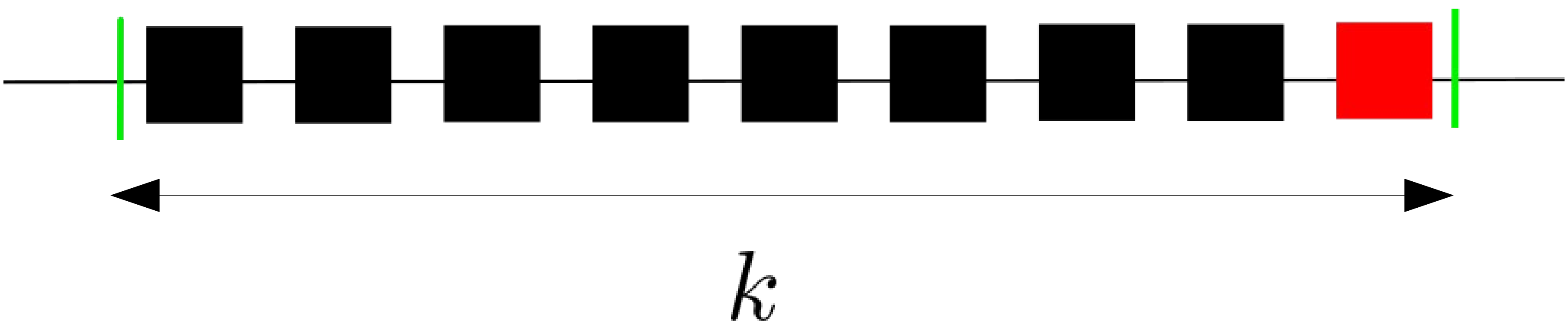

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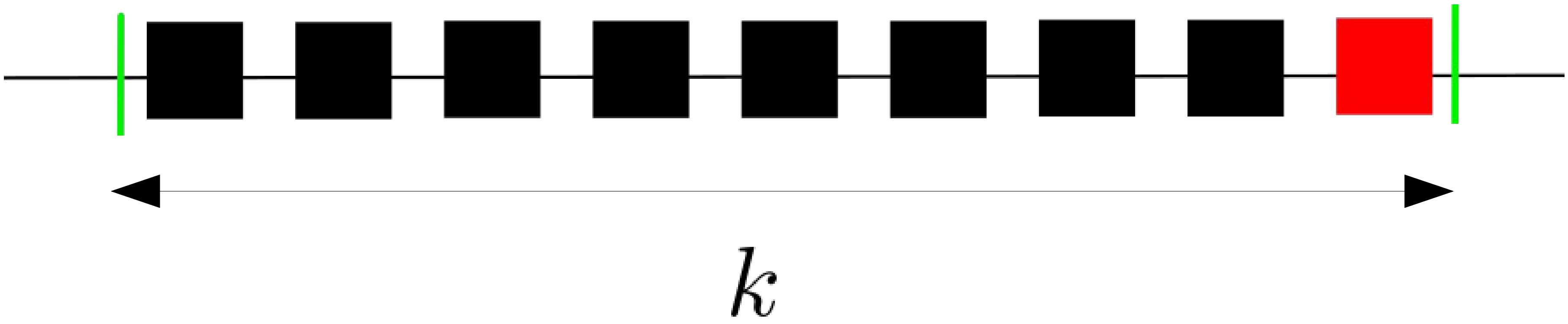

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EX: Symmetric random walk $\widehat{F}(s,k) = \frac{\sinh(\sqrt{2s}(k-1)) + \sinh(\sqrt{2s})}{\sinh(\sqrt{2s}k)}$

Step 1: We start by solving the exit time problem from an interval.

Step 2: We use the key equation,

$$\mathbb{P}\big(N(t_1) \geq n_1; N(t_2) \geq n_2\big) = \mathbb{P}\big(\tau_0 + ... + \tau_{n_1-1} \leqslant t_1; \tau_0 + ... + \tau_{n_2-1} \leqslant t_2\big)$$

Step 1: We start by solving the exit time problem from an interval.

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$$\begin{aligned}\mathcal{L}\{\mathbb{P}(N(t_1) \geq n_1, N(t_2) \geq n_2)\} &= \frac{1}{s_1 s_2}\widehat{F}(s_1+s_2,0)\dots\widehat{F}(s_1+s_2,n_1-1)\widehat{F}(s_2,n_1)\dots\widehat{F}(s_2,n_2-1)\\ &= \frac{1}{s_1 s_2}\exp\left(\sum_{k<n_1}\ln\widehat{F}(s_1+s_2,k) + \sum_{n_1\leq k<n_2}\ln\widehat{F}(s_2,k)\right)\end{aligned}$$

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EX: Symmetric random walk

$$\widehat{F}(s,k) = 1 - \sqrt{2s}\tanh\left(\sqrt{sk^2/2}\right) \qquad h(s,n) = \cosh^2\left(\sqrt{sn^2/2}\right)$$

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The generalization to the **k times distribution** is now straightforward, following the same three steps, valid for symmetric and correlated jumps,

$$\mathcal{L}\left\{\mathbb{P}\left(N(t_1) \geqslant n_1; \ldots; N(t_k) \geqslant n_k\right)\right\}$$

$$= \frac{1}{s_1 \ldots s_k} \frac{1}{h(s_1 + \ldots + s_k, n_1)} \frac{h(s_2 + \ldots + s_k, n_1)}{h(s_2 + \ldots + s_k, n_2)} \cdots \frac{h(s_k, n_{k-1})}{h(s_k, n_k)}$$

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Indeed, it does: There are long range correlations

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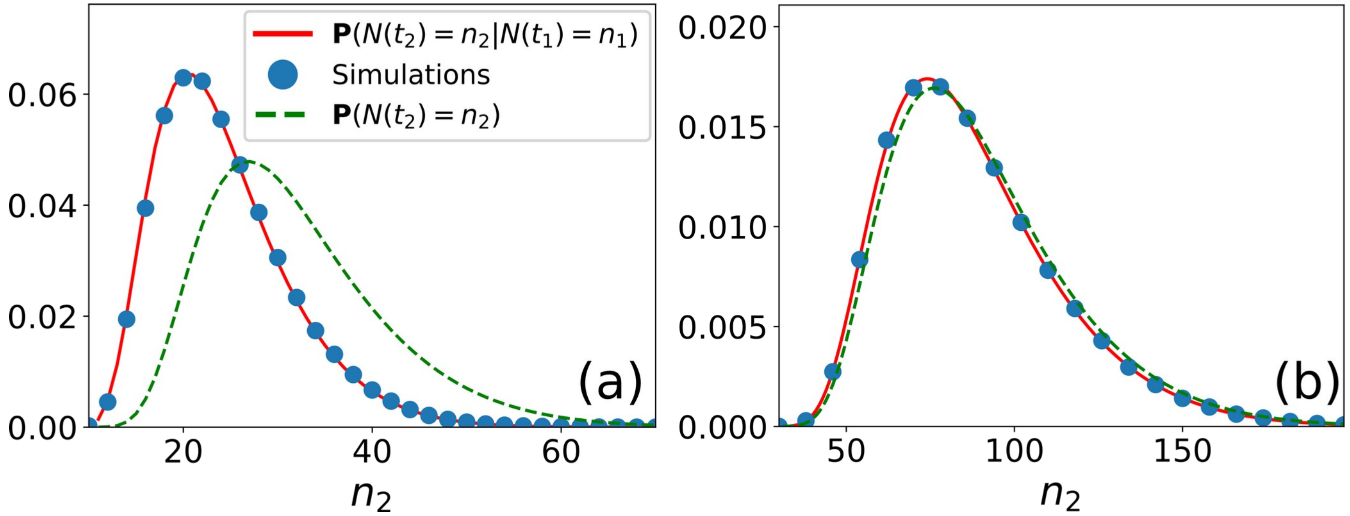

FIG: Conditional distribution for values t_1=200 and n_1=10 and (a) t_2=400 (b) t_2=3200

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It is not:

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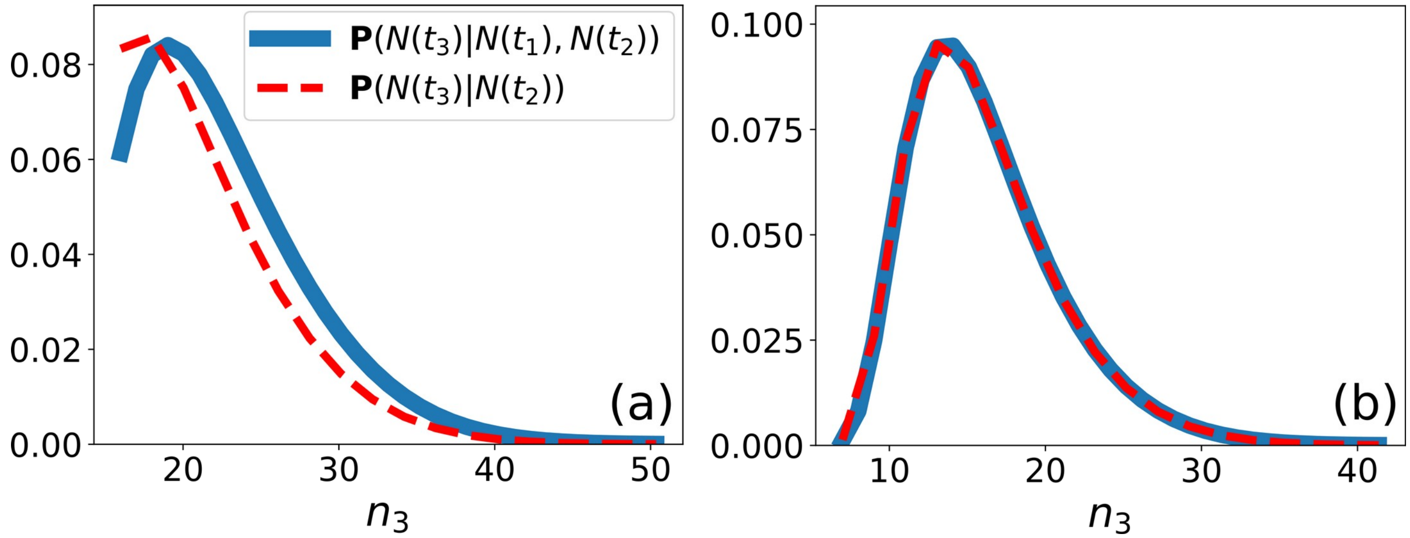

FIG: Distribution of N (t_3 = 200) conditioned on N (t_1 = 100) = 5 and (a) N (t_2 = 110) = 15 and (b) N (t_2 = 110) = 6 (blue curves); and distribution of N (t_3) conditioned only on N (t_2) (red).

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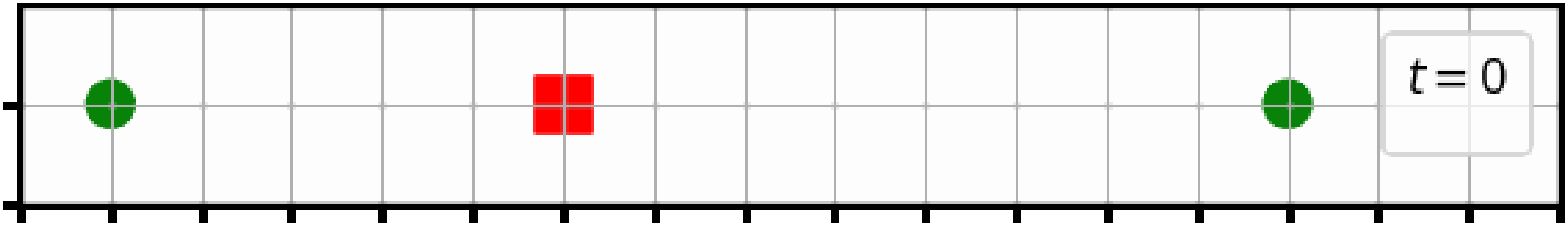

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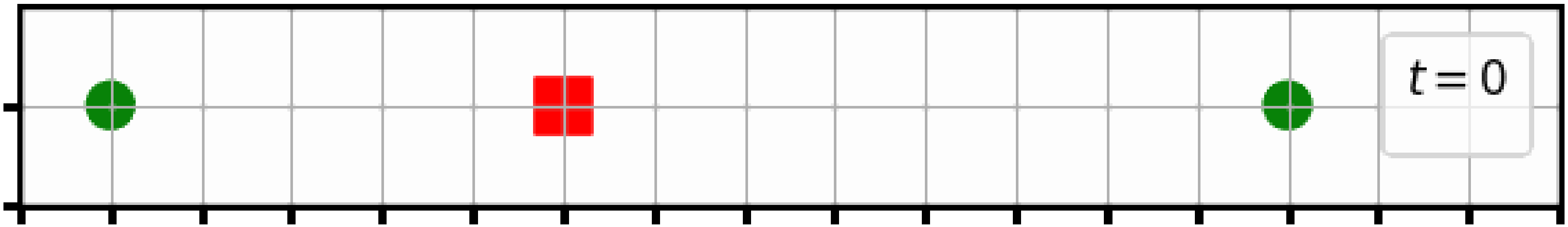

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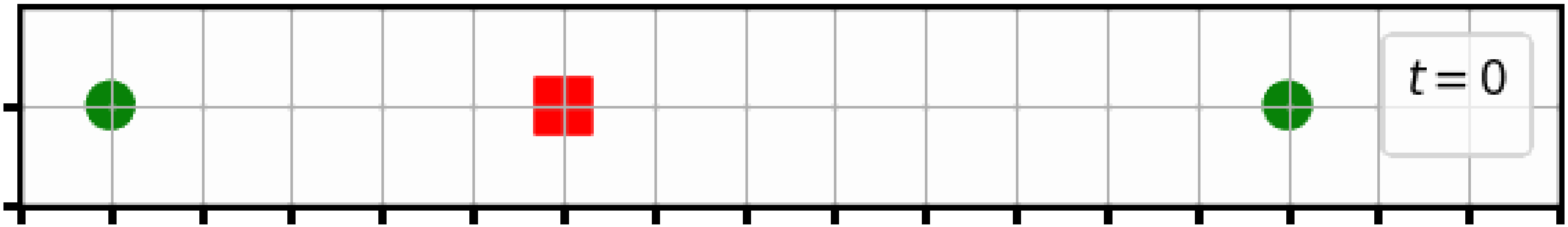

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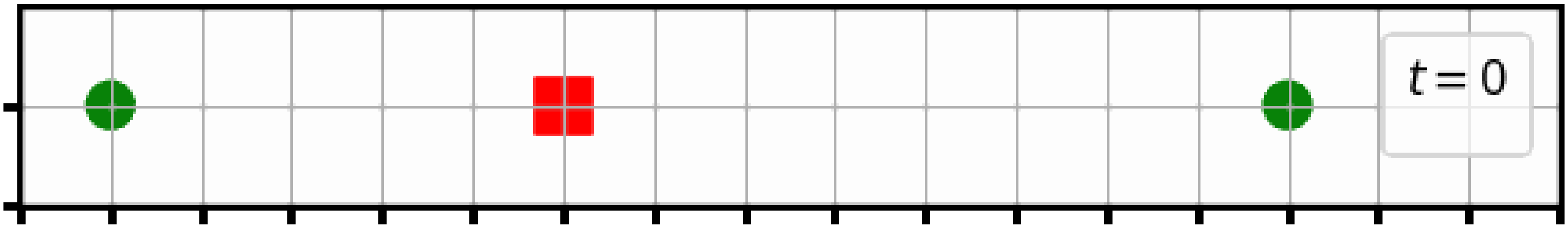

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$$S(t_2|t_1) = \langle (1-c)^{N(t_2)-N(t_1)} \rangle$$

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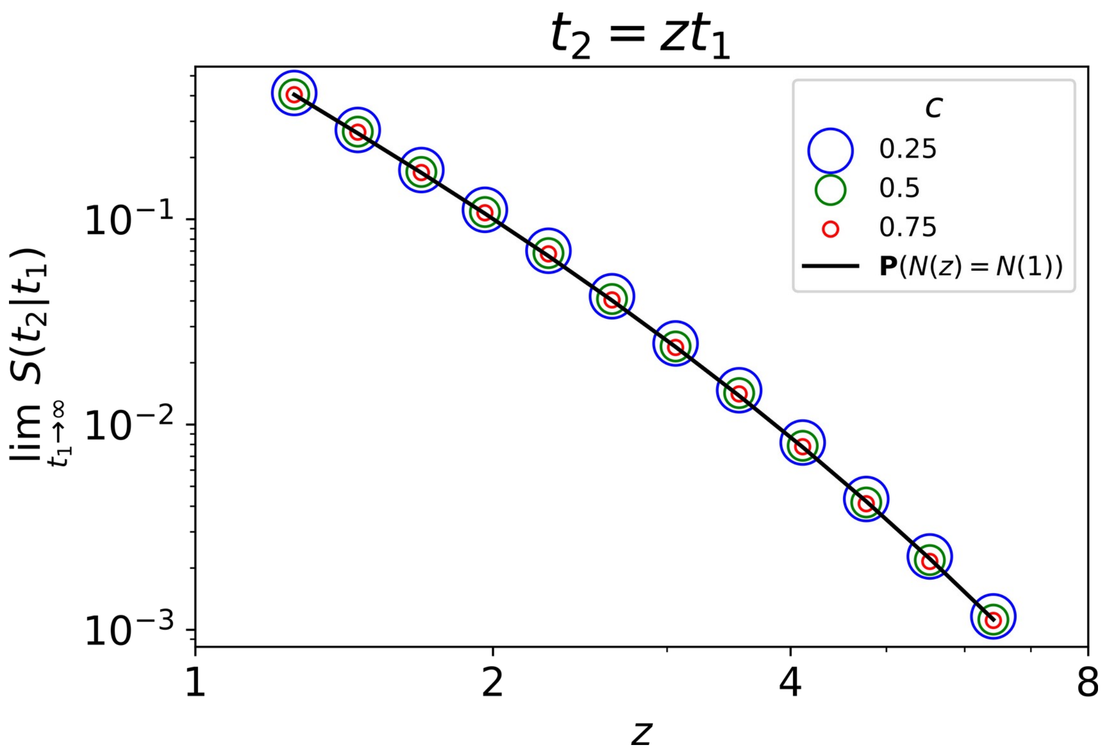

FIG: Conditional survival probability at large times in the limit $t_2=z\, t_1$, z fixed.

For the biased random walks, what about the multiple time distribution?

$$\mathbb{P}(N(t_1) = n_1, \ldots, N(t_k) = n_k)$$

We adapt the general steps to obtain the **cumulative two times distribution**, for a symmetric walk to the biased random walk,

$$\mathbb{P}(N(t_1) \geq n_1; N(t_2) \geq n_2)$$

Step 1: We start by solving the exit time problem from an interval, but now the position of start and matters.

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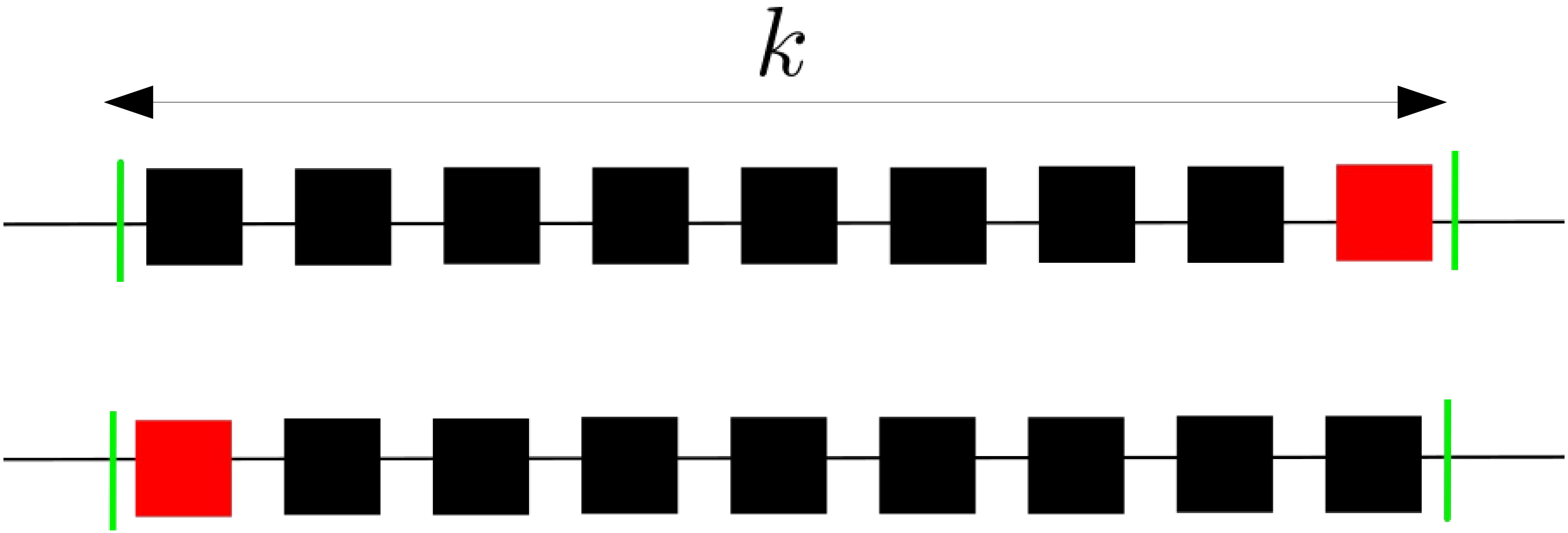

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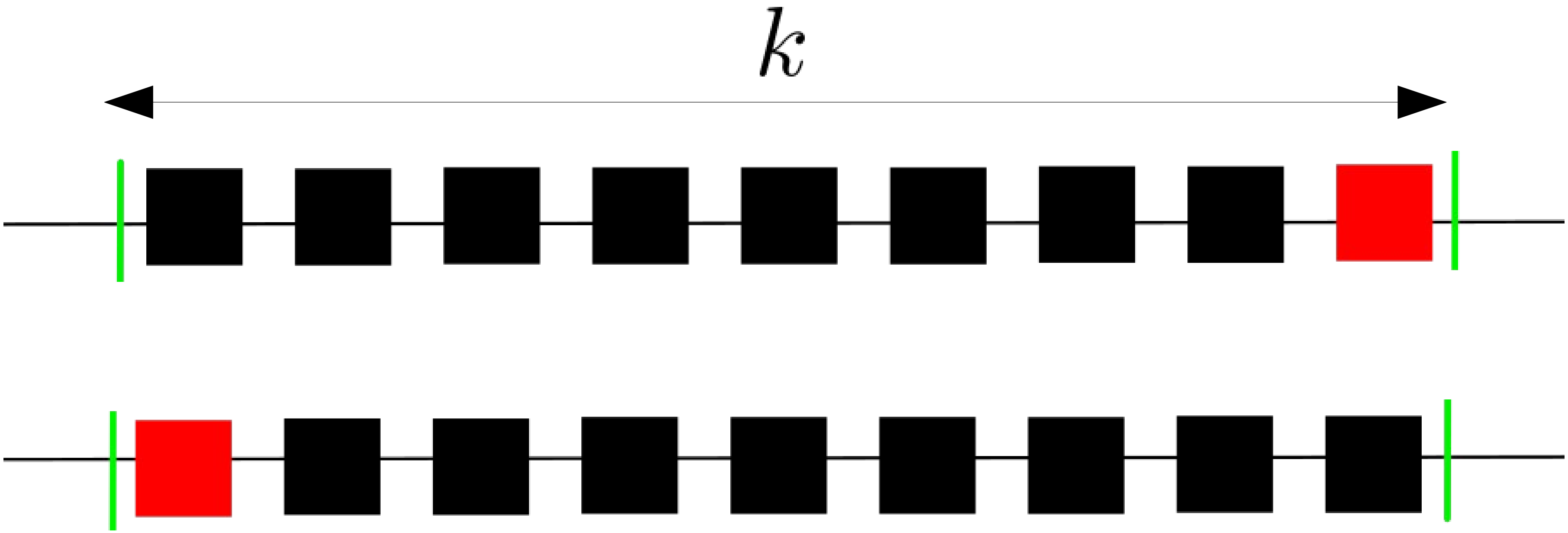
$$\widehat{F}(s,k) \equiv \begin{pmatrix} \widehat{F}^{l\to l}(s,k) & \widehat{F}^{l\to r}(s,k) \\ \widehat{F}^{r\to l}(s,k) & \widehat{F}^{r\to r}(s,k) \end{pmatrix}$$



Step 1: We start by solving the exit time problem from an interval, but now the position matters.

Step 2: We use the key equation, and Laplace transform it in both time variables using the independence of the τ_k's, but scalars are replaced by matrices.

$$\mathbb{P}\big(N(t_1) \geq n_1; N(t_2) \geq n_2\big) = \mathbb{P}\big(\tau_0 + ... + \tau_{n_1-1} \leqslant t_1; \tau_0 + ... + \tau_{n_2-1} \leqslant t_2\big)$$

$$\mathcal{L}\{\mathbb{P}(N(t_1) \geq n_1, N(t_2) \geq n_2)\} = \frac{1}{s_1 s_2}\widehat{F}(s_1+s_2,0)\dots\widehat{F}(s_1+s_2,n_1-1)\widehat{F}(s_2,n_1)\dots\widehat{F}(s_2,n_2-1)$$

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Step 3: We take the continuum limit.

We obtain in this limit $\widehat{F}(s,k) = 1 + g(s,k) \quad M(s,m,n) =: \exp\int_m^n g(s,k)dk :$

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The generalization to the **k times distribution** is now straightforward, following the three steps, valid for biased jumps,

$$\mathcal{L}\left\{\mathbb{P}(N(t_1) \geq n_1; \ldots; N(t_k) \geq n_k)\right\}$$

$$= \frac{1}{2s_1 s_2 \ldots s_k}\,(1,1) \times M(s_1 + \ldots + s_k, 0, n_1) M(s_2 + \ldots + s_k, n_1, n_2) \times \ldots \times M(s_k, n_{k-1}, n_k) \times (1,1)^T\,,$$

... and it can be used to obtain numerical values!

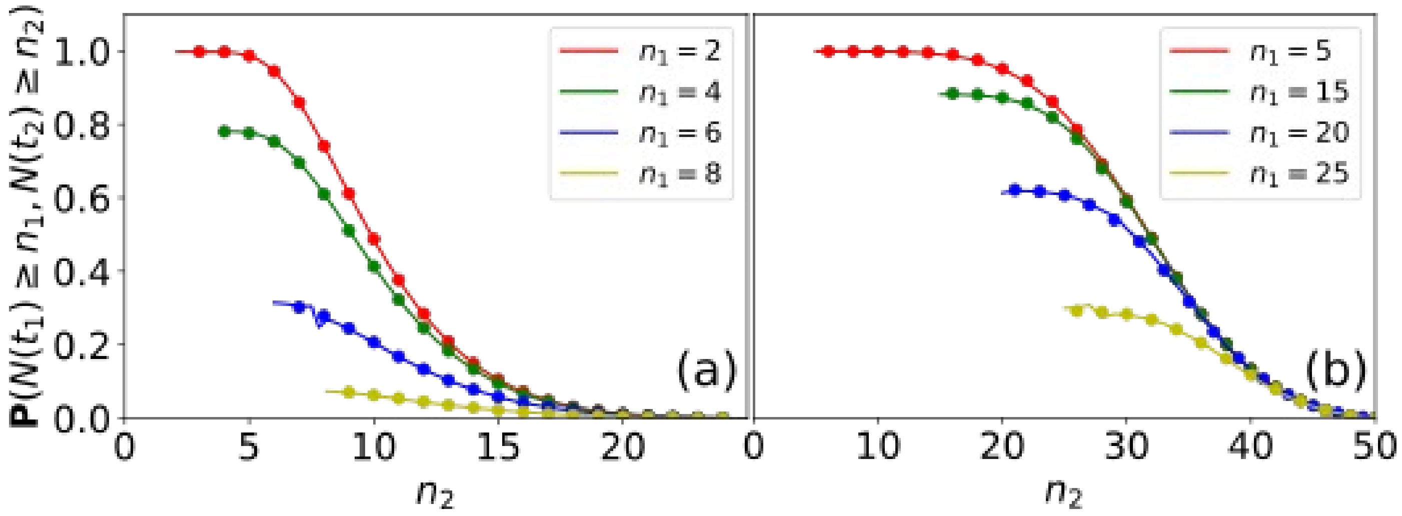

FIG : Numerical inversion of the two time distribution of the number of distinct sites visited for (a) persistent random walk (b) biased random walk

Check the link if you want more details!

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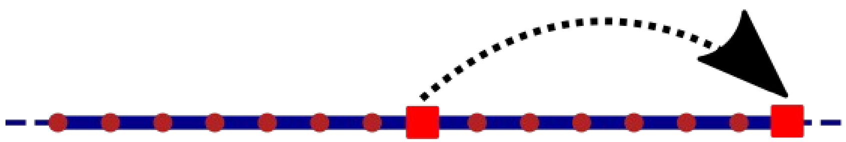
Any Questions?

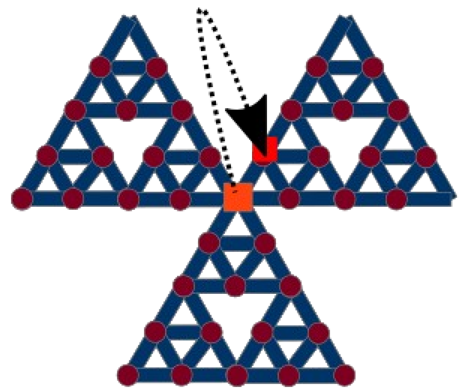
Generalisation:

1d Lévy flight

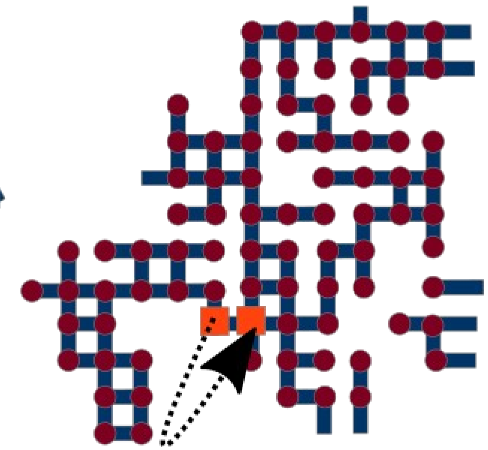


$$p(l) \propto l^{-1-\alpha}$$

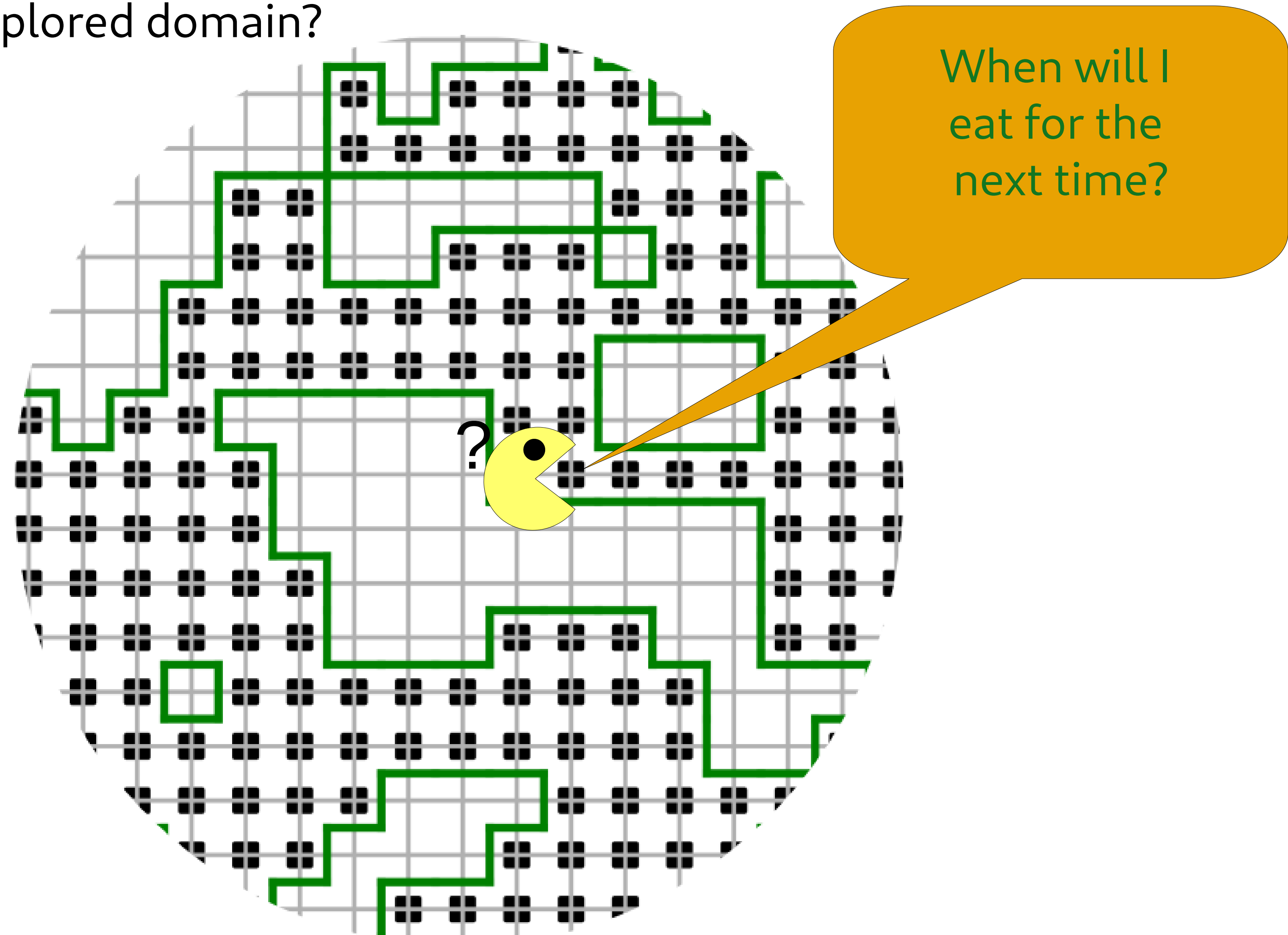
Sierpinski lattice



Percolation cluster



Question: Can we obtain $\mathcal{T}_n$ statistics in the case of more complex geometry of the explored domain?



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